# BRIEF REPORTS

## Power-law behavior in a nonextremal Bak-Sneppen model

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We propose a version of the Bak-Sneppen model in which the extremal dynamics is relaxed onto a stochastic dynamics, dependent on the fitness $f$. This system self-organizes into a stationary state, characterized by the absence of a sharp threshold in the fitness distribution and by nontrivial power law temporal correlations. The properties of these correlations are studied and discussed in detail, and the behavior of the fitness distribution is studied analytically. [S1063-651X(98)02009-1]

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Self-organized criticality (SOC) has been widely investigated during the past few years, as one of the prominent candidates to explain the spontaneous critical behavior of many natural phenomena [1,2]. Systems exhibiting SOC behavior are characterized by the absence of any time or length scale, resulting in a *power law* distribution of many relevant quantities [3–6]. From the theoretical point of view, SOC behavior emerges in many models with *extremal dynamics* in which the element that has extremal properties drives the evolution of the system [2]. Typical examples of such models are invasion percolation for fluid displacement in porous media [5], and the Sneppen model for the dynamics of surfaces pinned by quenched disorder [4]. The Bak-Sneppen (BS) model of biological evolution is the simplest extremal model realizing SOC [3].

In its original version, the BS model describes a one-dimensional ''food chain'' of $N$ sites (''species'') in which the fitness $f$ of each species is described by a random number between 0 and 1. At every time step the site with the smallest fitness (in the following referred to as the active site) is found, and its fitness (as well as the fitnesses of its two nearest neighbors) changed at random. The critical state is a stable attractor of the dynamics, and is characterized by a threshold in the population density, and by power law distributions of some characteristic quantities such as the avalanche size distribution and the first- and all-return times distributions of the activity [3].

A question that arises naturally is to what extent the application of extremal dynamics is justified in the biological context. Extremal dynamics emerges from the large separation of time scales typical of evolution in fitness landscapes: Species that are in a local minimum of the fitness landscape evolve toward a nearby peak much faster than highly fit species drift away from their privileged condition [7–9]. One could, however, think of situations in which several species

are in local minima within the same ecological network. For these species the separation of time scales can be relaxed to allow more than one species to mutate, i.e., every species has a chance to be the next one to evolve, but the least fit species is still the most likely to change at the next time step. In this paper we relax the extremal dynamics of the BS model according to this principle. Every species is assigned a probability $p(f)$ to be selected that depends on its fitness. We have chosen the class of $p(f)$ spanned by a parameter $\alpha$ according to

$$p_\alpha(f) = \left( \sum_{i=1}^{N} f_i^{-\alpha} \right)^{-1} f^{-\alpha}, \qquad (1)$$

where $N$ is the number of sites (species) on the lattice. The BS model corresponds to the limit $\alpha \to \infty$.

A relaxation of the extremal rule has already been introduced in the BS model in the way of a temperaturelike parameter [10]. In that case, the probability of a site to be chosen as active was given by a Boltzmann weight factor $p_T(f) \sim e^{-f/T}$, that upon normalization gives

$$p_T(f) = \left( \sum_{i=1}^{N} e^{-f_i/T} \right)^{-1} e^{-f/T}. \qquad (2)$$

On introducing probability (2), one sees that the model is critical only in the limit $T \to 0$. Any value of temperature $T > 0$ leads to a different behavior [10]. Our scheme is more in line with recent results in the context of polymers in a disordered environment, where choices such as Eq. (1) are introduced in order to move from the weak disorder limit to the strong disorder case. In that context it is observed that the strong disorder regime emerges only for $\alpha = \infty$, $\alpha$ finite being always in the weak disorder universality class [12]. Our results show a similar trend, but in addition some nontrivial
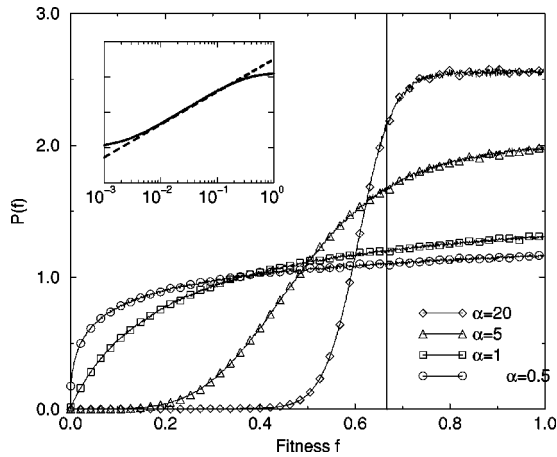
FIG. 1. Fitness distribution for different values of $\alpha$. The vertical line represents the position of the threshold of the Bak-Sneppen model. In the inset (where the units are the same as in the large figure) we show a power law fit (dashed line) of the fitness probability distribution $n(f)$ for $\alpha = 2$ (solid line). The fit yields an exponent $\alpha' = 1.85$, that is indeed very close to the mean field prediction $\alpha' = 2$. The deviation from power law behavior for very small $f$ in $n(f)$ is due to finite size effects.

features appear. In particular, although the system is always noncritical for any $\alpha < \infty$, some quantities exhibit a power law behavior at intermediate times for $\alpha > 1$. At variance with the approach of Ref. [10], defining the probabilities via Eq. (1) ensures that when the fitness of a site tends to zero, the probability of choosing it approaches one (independent of the lattice size), that is $p_\alpha(f \to 0) \to 1$. In the case of $p_T$, on the other hand, the same limit yields a number that approaches zero as $N \to \infty$.

Let us now proceed to discuss the results obtained for the probability $p_\alpha(f)$ [for simplicity, we will refer to this as $p(f)$ in the future]. The fitness probability distribution $n(f)$ at stationarity is plotted in Fig. 1, for different values of $\alpha$. For large $\alpha$ the distribution goes to the BS step function with a threshold at $f = f_c \approx 0.667$. However, for any finite $\alpha$ there remains a tail of nonzero occupation probability, even at small fitnesses. This is what we naively expect, since any fitness has a certain probability to survive.

The qualitative behavior of $n(f)$ can be derived from a mean field master equation (spatial correlations are neglected)

$$\dot{n}(f,t) = \int_0^1 P(y \to f)n(y,t)dy + 2\int_0^1 n(y,t)dy$$
$$- \int_0^1 P(f \to y)n(f,t)dy - 2n(f,t)\int_0^1 dy. \quad (3)$$

In this equation, $P(a \to b)$ represents the probability that the fitness $a$, of the chosen species, is changed to $b$. The two terms not containing $P(a \to b)$ give the contribution of the neighbors. In our model we have $P(y \to f) = M(\alpha)y^{-\alpha}$ and $P(f \to y) = M(\alpha)f^{-\alpha}$, with $M(\alpha)$ given by the normalization. From Eq. (1) we see that, in the limit $N \to \infty$,

$$M(\alpha) = \left[\int_0^1 n(f)f^{-\alpha}df\right]^{-1}. \quad (4)$$

This relation implies that for $\alpha > 1$, $n(f) \sim f^{\alpha'}$ for $f \to 0$, with $\alpha' \geq \alpha - 1$. By inserting these relations into Eq. (3), we obtain

$$\dot{n}(f,t) = M(\alpha)\int_0^1 dy \frac{n(y,t)}{y^\alpha} + 2 - M(\alpha)\int_0^1 dy \frac{n(f,t)}{f^\alpha}$$
$$- 2n(f,t)$$
$$= 3 - M(\alpha)\frac{n(f,t)}{f^\alpha} - 2n(f,t). \quad (5)$$

In the stationary state, $n(f,t)$ does not depend on time, and therefore the solution of Eq. (5) reads

$$n(f) = \frac{3f^\alpha}{2f^\alpha + M(\alpha)}. \quad (6)$$

Since $\alpha' = \alpha$, the normalization condition is fulfilled. Using the normalization condition together with Eq. (6), it is possible to show that $M(\alpha) \sim 2 \times 3^{-\alpha}$ for $\alpha \to \infty$. Substituting this expression back into Eq. (6), we recover, in the limit $\alpha \to \infty$, the mean field fitness distribution of the Bak-Sneppen model, namely, $n(f) = \frac{3}{2}\theta(f - \frac{1}{3})$. Turning our attention once again to Fig. 1, in the inset one can see that, for $\alpha = 2$, the exponent $\alpha'$ obtained numerically is very close to the mean field prediction.

We now come to the investigation of correlations within the asymptotic state. For this purpose we performed numerical simulations for values of $\alpha$ ranging from 0 to 50, and lattice sizes up to $2^{20}$ (the limitations introduced by the numerical treatment precluded us from going beyond $\alpha = 50$). In our simulations we collected statistics for avalanches and all- and first-return time distributions. The dynamics of our model not being extremal, the causal connection inside an avalanche cannot be defined in terms of a critical threshold $f_c$ for the fitness distribution (in fact, as Fig. 1 indicates, this threshold does not exist). One needs another definition of causal connection inside an avalanche. The one we adopt here has been tested in both extremal and nonextremal dynamics with quenched disorder [11].

The first return time distribution $P_f(t)$ is characterized by two different power law regimes. For short times, a behavior resembling that of the original BS model appears ($\tau_f = \tau_f^{(BS)}$). For longer times, the random choice of the active site over the lattice plays an important role in changing the value of the exponent to a new $\tau_f(\alpha)$. The exponents corresponding to this second regime are shown in Fig. 2, where one can see that $\tau_f(\alpha) \to 1^-$ as $\alpha$ increases. The all-return time distribution exhibits the same kind of crossover: The exponents $\tau_a(\alpha)$ corresponding to the second regime are also shown in Fig. 2. Our simulations indicate that the value of $t$ at which the crossover takes place increases as $\alpha$ increases. This reflects the fact that the system behaves more and more like the BS model as $\alpha \to \infty$. We will come back to this point in the case of the avalanches.

To understand better the behavior of the system away from the BS regime, we proceed to analyze in detail the first-return time distribution in the case $\alpha = 2$. In this case, the BS behavior is relegated to time scales comparable with the microscopic times and, therefore, of no relevance. More-

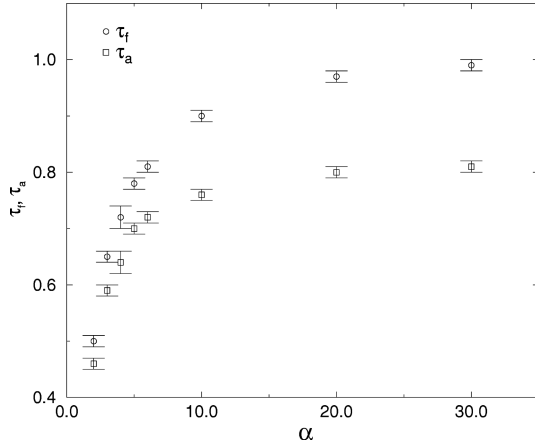FIG. 2. First-return time exponent (circles) and all-return time exponents (squares) vs $\alpha$.

over, the fact that the exponent is less than one (see Fig. 2), implies the presence of a cutoff in the distribution which depends on the system size $N$. As we show below, the dependence of the cutoff on $N$ can be used to understand the underlying processes leading to a non-BS power law behavior.

In Fig. 3, the first-return distribution $P_f(t,N)$ is plotted for $\alpha=2$ and different system sizes $N=2^{12}$, $2^{14}$, and $2^{16}$. One can clearly see a power law behavior with an exponent $\tau_f(\alpha=2)=0.50(1)$, followed by an exponential cutoff related to the finite size of the system. In the inset in Fig. 3, we plot the collapse of the same data according to the law

$$P_f(t,N) = t^{-\tau_f} N^{-z(1-\tau_f)} f\left(\frac{t}{N^z}\right). \qquad (7)$$

The data for different systems sizes lay one on each other with the choices $\tau_f(\alpha=2)=0.50(1)$ and $z=1$. The exponent $z=1$ is the same as for a completely random choice of the active site over the lattice (corresponding to $\alpha=0$), where the first-return probability can be exactly computed as
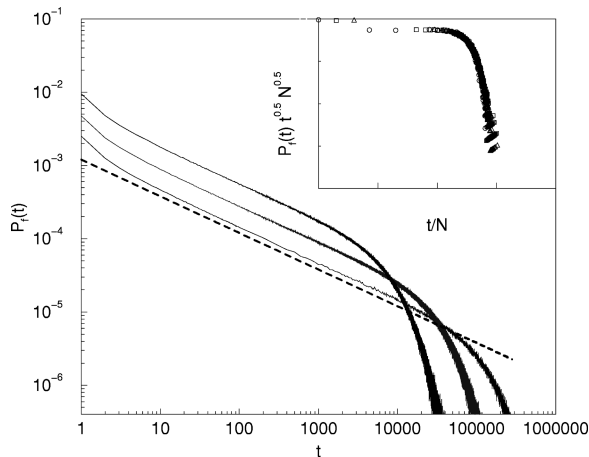


FIG. 3. First-return time probability distribution with an exponent $\alpha=2$ and different system sizes $N=2^{12}$, $2^{14}$, and $2^{16}$. The dashed line is a power law with exponent 0.50. The collapse plot of the three distributions according to Eq. (7) is shown in the inset. Both the main figure and the inset are plotted on a $\log_{10}$-$\log_{10}$ scale. $t$ is measured in time steps.


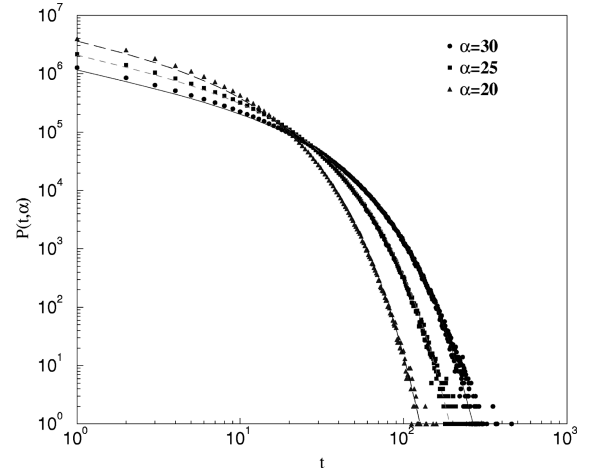
FIG. 4. This figure shows the unnormalized avalanche probability distributions for different values of $\alpha$. The different lines represent the best fit given by Eq. (9). The figure is plotted on a $\log_{10}$-$\log_{10}$ scale, and $t$ is measured in time steps.

$$P_f(t,N) = \frac{1}{N}\left(1-\frac{1}{N}\right)^{t-1} \sim \frac{1}{N} e^{-t/N}, \quad N \gg 1. \qquad (8)$$

This shows that the cutoff in the $P_f(t,N)$ is not a finite size effect but, rather, evidence of noncriticality. The presence of a damping factor $N^{-z(1-\tau_f)}$ in Eq. (7) is necessary for the normalization of the first-return probability in the thermodynamic limit.

The fact that the system is not critical can be better understood by looking at the avalanche distributions. As can be seen in Fig. 4, a nonintegrable power law behavior is followed by an exponential decay. The best fit for the avalanche distributions is provided by the function

$$P(t,\alpha,N) \sim t^{-\tau} e^{-\lambda(\alpha,N)t}, \qquad (9)$$

where $\lambda^{-1}(\alpha,N)$ is the characteristic length scale for the avalanches, and $\tau=0.60(2)$ for any value of $\alpha$. We also computed the values of $\lambda$ for different $\alpha$'s and different values of $N$. The results are shown in Fig. 5, and lead to $\lambda$
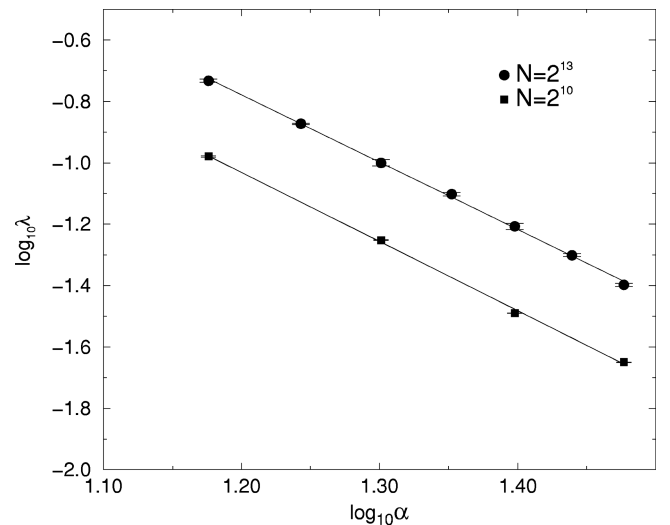


FIG. 5. Scaling of the characteristic time $\lambda(\alpha)$ vs $\alpha$ ($15<\alpha<30$) (see text for details).

$=A(N)\alpha^{-\delta}$, where $A(N=2^{13})=71(2)$, $A(N=2^{10})=43(2)$ and $\delta=2.2(2)$, and $\delta$ independent of $N$.

In this paper we have introduced a version of the Bak-Sneppen model in which a stochastic dynamics replaces the original extremal one. The present modification of the BS model is noncritical, just as in previous attempts to relax extremal dynamics [10]. Nonetheless, at variance with Ref. [10], we have shown that it is possible to recover a BS-like behavior for first- and all-return distributions (on small time scales) for large values of $\alpha$. The long-time scales also exhibit nontrivial power law correlations, whose origin can be understood in terms of the random choice of the minimum due to early end of the avalanches.

Further insights can be obtained by studying the distribution of the avalanches. On the one hand, this allows us to compute the typical time length $\lambda^{-1}$ introduced into the system with the parameter $\alpha$. On the other hand, it leaves us with the unsolved problem of understanding the independence of the exponent $\tau$ on $\alpha$, evidence that contradicts the naive expectation and the behavior of the other distributions studied here. It is worth pointing out that an analogous intermediate time behavior was recently observed in the integration of population dynamics equations (such as Lotka-Volterra equations) [13]. This work can be considered as an attempt to build a bridge between extremal models (showing SOC behavior) and continuous equations (e.g., Lotka-Volterra equations), the most frequently used paradigms in biological contexts.

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